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Inertia theorems for pairs of matrices ☆

Cristina Ferreira [a], Fernando C. Silva [b,*]

[a] *Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal*

[b] *Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, Rua Ernesto de Vasconcelos, Campo Grande, 1749-016 Lisbon, Portugal*

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Abstract

Let L be a square matrix. A well-known theorem due to Lyapunov states that L is positive stable if and only if there exists a Hermitian positive definite matrix H such that $LH + HL^*$ is positive definite. The main inertia theorem, due to Ostrowski, Schneider and Taussky, states that there exists a Hermitian matrix H such that $LH + HL^*$ is positive definite if and only if L has no eigenvalues with zero real part; and, in that case, the inertias of L and H coincide.

A pair (A, B) of matrices of sizes $p \times p$ and $p \times q$, respectively, is said to be positive stabilizable if there exists X such that $A + BX$ is positive stable. In this paper, we generalize Lyapunov's theorem by giving necessary and sufficient conditions for (A, B) being positive stabilizable. We also give generalizations of the main inertia theorem and of another inertia theorem due to Chen and Wimmer. Then we deduce a necessary condition for the existence of a Hermitian matrix H such that $K := LH + HL^*$ is positive semidefinite and the number of nonconstant invariant factors of $\left[xI - L \;\middle|\; K\right]$ has a fixed value. This last result was inspired by another inertia theorem due to Loewy.

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* Corresponding author.

E-mail addresses: cristina@mat.ua.pt (C. Ferreira), fcsilva@fc.ul.pt (F.C. Silva).

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1. Lyapunov's theorem

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of the real and of the complex numbers, respectively. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The *inertia* of a polynomial $f \in \mathbb{F}[x]$ is the triple $\text{In}(f) = (\pi(f), \nu(f), \delta(f))$, where $\pi(f)$, $\nu(f)$ and $\delta(f)$ denote, respectively, the number of roots of f with real positive part, the number of roots of f with real negative part and the number of roots of f with real part equal to zero. The *inertia* of $L \in \mathbb{F}^{n\times n}$, denoted by $\text{In}(L) = (\pi(L), \nu(L), \delta(L))$, is the inertia of its characteristic polynomial, $\det(xI_n - L)$; L is said to be *positive stable* if $\text{In}(L) = (n, 0, 0)$. As usual, given a square matrix H, the expression $H > 0$ means that H is Hermitian positive definite and $H \geqslant 0$ means that H is Hermitian positive semidefinite.

Lyapunov's theorem states that $L \in \mathbb{F}^{n\times n}$ is positive stable if and only if there exists a positive definite matrix $H \in \mathbb{F}^{n\times n}$ such that $LH + HL^* > 0$. A generalization, due to Ostrowski and Schneider [5] and to Taussky [8], states that there exists a Hermitian matrix $H \in \mathbb{F}^{n\times n}$ such that $LH + HL^* > 0$ if and only if $\delta(L) = 0$; and the inequality $LH + HL^* > 0$ also implies that $\text{In}(L) = \text{In}(H)$.

A linear system $\dot{x}(t) = Ax(t) + Bu(t)$, where $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$ and $u(t)$ is a control vector, is said to be *stabilizable* if there exists a linear feedback input $u(t) = Xx(t)$, with $X \in \mathbb{F}^{q\times p}$, such that the system becomes stable, that is, $A + BX$ is (negative) stable. The pair (A, B), where $A \in \mathbb{F}^{p\times p}$ and $B \in \mathbb{F}^{p\times q}$, is said to be *positive stabilizable* if there exists $X \in \mathbb{F}^{q\times p}$ such that $A + BX$ is positive stable.

The description of the possible characteristic polynomials of $A + BX$, when X varies, presented in the next lemma, is a well-known result in control theory, see [7, Theorem 13] for example. See [14, Theorem 2.6] for a more general result.

Lemma 1. *Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$. Let $f \in \mathbb{F}[x]$ be a monic polynomial of degree p. There exists $X \in \mathbb{F}^{q\times p}$ such that $A + BX$ has characteristic polynomial f if and only if the product of the invariant factors of*

$$\begin{bmatrix} xI_p - A & \big| & B \end{bmatrix} \tag{1}$$

divides f.

If $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$, define the *inertia* of (A, B) as $\text{In}(A, B) = (\pi(A, B), \nu(A, B), \delta(A, B))$, where $\pi(A, B)$, $\nu(A, B)$, $\delta(A, B)$ denote, respectively, the number of roots of the product of the invariant factors of (1) with real positive part, real negative part and real part equal to zero. For notational convenience, we make convention that the invariant factors of polynomial matrices are always monic.

Lemma 2 [10]. *Let $A \in \mathbb{F}^{p\times p}$ and $B \in \mathbb{F}^{p\times q}$. Let $f \in \mathbb{F}[x]$ be a monic polynomial of degree $p + q$. There exist $C \in \mathbb{F}^{q\times p}$, $D \in \mathbb{F}^{q\times q}$ such that*

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2}$$

has characteristic polynomial f if and only if the product of the invariant factors of (1) *divides f.*

Given two triples of integers (π, ν, δ) and (π', ν', δ'), we shall write $(\pi, \nu, \delta) \leqslant (\pi', \nu', \delta')$ whenever $\pi \leqslant \pi'$, $\nu \leqslant \nu'$ and $\delta \leqslant \delta'$.

Remark. Let $A \in \mathbb{F}^{p\times p}$ and $B \in \mathbb{F}^{p\times q}$. The following statements are trivial corollaries of Lemmas 1 and 2:

(1) (A, B) is positive stabilizable if and only if the roots of the product of the invariant factors of (1) have positive real parts.
(2) (A, B) is positive stabilizable if and only if there exist $C \in \mathbb{F}^{q\times p}$, $D \in \mathbb{F}^{q\times q}$ such that (2) is positive stable.
(3) Given a matrix of the form (2), $\mathrm{In}(A, B) \leqslant \mathrm{In}(L)$.

Now Lyapunov's theorem can be easily extended to pairs of matrices as the following theorem shows.

Theorem 3. *Let $A \in \mathbb{F}^{p\times p}$ and $B \in \mathbb{F}^{p\times q}$. The following are equivalent*:

(a_3) *The pair (A, B) is positive stabilizable.*
(b_3) *There exists a positive definite matrix*

$$H = \begin{bmatrix} H_1 & H_2 \\ * & * \end{bmatrix} \in \mathbb{F}^{(p+q)\times(p+q)}, \quad \text{where } H_1 \in \mathbb{F}^{p\times p}, \tag{3}$$

such that

$$AH_1 + H_1A^* + BH_2^* + H_2B^* > 0. \tag{4}$$

(c_3) *There exists a positive definite matrix $H_1 \in \mathbb{F}^{p\times p}$ and there exists $H_2 \in \mathbb{F}^{p\times q}$ such that* (4) *is satisfied.*

Proof. (a_3) *implies* (b_3). As we have already noticed, there exist $C \in \mathbb{F}^{q\times p}$, $D \in \mathbb{F}^{q\times q}$ such that (2) is positive stable. According to Lyapunov's theorem, there exists a positive definite matrix $H \in \mathbb{F}^{(p+q)\times(p+q)}$ such that $LH + HL^* > 0$. Partition H as in (3). Then $AH_1 + H_1A^* + BH_2^* + H_2B^*$ is a principal submatrix of $LH + HL^*$ and, therefore, is positive definite.

(b_3) *implies* (c_3). Trivial.

(c_3) *implies* (a_3). From (4) it follows that

$$(A + BH_2^*H_1^{-1})H_1 + H_1(A^* + H_1^{-1}H_2B^*) > 0.$$

According to Lyapunov's theorem, $A + BH_2^*H_1^{-1}$ is positive stable. Therefore (A, B) is positive stabilizable. $\square$

2. The inertia theorem of Ostrowski, Schneider and Taussky

The second natural problem is to extend the inertia theorem, due to Ostrowski, Schneider [5] and Taussky [8], to pairs of matrices. Note that, for every $L \in \mathbb{F}^{n\times n}$, every Hermitian matrix $H \in \mathbb{F}^{n\times n}$ and every nonsingular matrix $S \in \mathbb{F}^{n\times n}$, $LH + HL^* > 0$ if and only if $(SLS^{-1})(SHS^*) + (SHS^*)(SLS^{-1})^* > 0$. Then that inertia theorem can be viewed as giving a complete set of relations between the similarity orbit of L and the congruence orbit of H, when $LH + HL^* > 0$; and can be stated as follows.

Theorem 4. *Let $L, H \in \mathbb{F}^{n\times n}$, where H is Hermitian. The following are equivalent*:

(a_4) *There exists a Hermitian matrix $H' \in \mathbb{F}^{n\times n}$, congruent to H, such that $LH' + H'L^* > 0$.*
(b_4) *There exists $L' \in \mathbb{F}^{n\times n}$, similar to L, such that $L'H + HL'^* > 0$.*
(c_4) *There exists a Hermitian matrix $H' \in \mathbb{F}^{n\times n}$, congruent to H, and there exists $L' \in \mathbb{F}^{n\times n}$, similar to L, such that $L'H' + H'L'^* > 0$.*
(d_4) $\delta(L) = 0$ *and* $\mathrm{In}(L) = \mathrm{In}(H)$.

Now let $A, H_1 \in \mathbb{F}^{p\times p}$, $B, H_2 \in \mathbb{F}^{p\times q}$, where H_1 is Hermitian. Then, for every nonsingular matrix

$$S = \begin{bmatrix} P & 0 \\ R & Q \end{bmatrix} \in \mathbb{F}^{(p+q)\times(p+q)}, \quad \text{where } P \in \mathbb{F}^{p\times p}, \tag{5}$$

(4) is equivalent to any of the following inequalities:

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} H_1 \\ H_2^* \end{bmatrix} + \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} A^* \\ B^* \end{bmatrix} > 0,$$

$$\left(P \begin{bmatrix} A & B \end{bmatrix} S^{-1}\right) \left(S \begin{bmatrix} H_1 \\ H_2^* \end{bmatrix} P^*\right) + \left(P \begin{bmatrix} H_1 & H_2 \end{bmatrix} S^*\right) \left((S^*)^{-1} \begin{bmatrix} A^* \\ B^* \end{bmatrix} P^*\right) > 0.$$

Recall that $[A \quad B]$ and $[A' \quad B']$, where $A, A' \in \mathbb{F}^{p\times p}$ and $B, B' \in \mathbb{F}^{p\times q}$, are said to be *block similar* or *feedback equivalent* if there exists a nonsingular matrix of the form (5) such that

$$\begin{bmatrix} A' & B' \end{bmatrix} = P \begin{bmatrix} A & B \end{bmatrix} S^{-1}.$$

It is easy to see that $[A \quad B]$ and $[A' \quad B']$ are block similar if and only if the linear pencils $\begin{bmatrix} xI_p - A \mid B \end{bmatrix}$ and $\begin{bmatrix} xI_p - A' \mid B' \end{bmatrix}$ are strictly equivalent. Then a canonical form for block similarity results easily from the Kronecker canonical form for strict equivalence. (See [3], for details about strict equivalence.)

Given a polynomial $f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$, with $k \geqslant 1$, over $\mathbb{F}$, denote by $C(f)$ the companion matrix of f,

$$C(f)=\left[\begin{array}{c|c}0 & I_{k-1}\\ \hline -a_0 & -a_1\cdots-a_{k-1}\end{array}\right]\in\mathbb{F}^{k\times k}.$$

Denote by $e_j^{(p)}$ the jth column of the identity matrix of order p, I_p.

Lemma 5 (Canonical form for block similarity [12, Theorem 2.11]). *Let $A\in\mathbb{F}^{p\times p}$, $B\in\mathbb{F}^{p\times q}$. The matrix $[A\quad B]$ is block similar to a unique matrix of the form*

$$\left[\begin{array}{cc|c}N & 0 & 0\\ 0 & M_1 & M_2\end{array}\right], \tag{6}$$

where

$$\begin{aligned}N&=C(f_1)\oplus\cdots\oplus C(f_w)\in\mathbb{F}^{d\times d},\\ M_1&=C(x^{\mu_1})\oplus\cdots\oplus C(x^{\mu_u})\in\mathbb{F}^{(p-d)\times(p-d)},\\ M_2&=\begin{bmatrix}e_{\mu_1}^{(p-d)} & e_{\mu_1+\mu_2}^{(p-d)} & \cdots & e_{\mu_1+\cdots+\mu_u}^{(p-d)} & 0\end{bmatrix}\in\mathbb{F}^{(p-d)\times q},\end{aligned}$$

$f_1(x)|\cdots|f_w(x)$, $w\geqslant 0$, $d=\deg(f_1\cdots f_w)$, $0<\mu_1\leqslant\cdots\leqslant\mu_u$, $u\geqslant 0$. *The polynomials $f_1,\ldots,f_w$ are the nonconstant invariant factors and $\mu_1,\ldots,\mu_u$ are the nonzero column minimal indices of* (1).

We shall say that two matrices $[H_1\quad H_2]$ and $[H_1'\quad H_2']$, where $H_1, H_1'\in\mathbb{F}^{p\times p}$ are Hermitian and $H_2, H_2'\in\mathbb{F}^{p\times q}$, are *block congruent* if there exists a nonsingular matrix of the form (5) such that

$$\begin{bmatrix}H_1' & H_2'\end{bmatrix}=P\begin{bmatrix}H_1 & H_2\end{bmatrix}S^*.$$

Lemma 6 (Canonical form for block congruence). *Let $H_1\in\mathbb{F}^{p\times p}$ be a Hermitian matrix and $H_2\in\mathbb{F}^{p\times q}$. Then $[H_1\quad H_2]$ is block congruent to a unique matrix of the form*

$$\left[\begin{array}{cccc|cc}I_\pi & 0 & 0 & 0 & 0 & 0\\ 0 & -I_\nu & 0 & 0 & 0 & 0\\ 0 & 0 & 0_\rho & 0 & I_\rho & 0\\ 0 & 0 & 0 & 0_{p-\pi-\nu-\rho} & 0 & 0\end{array}\right]. \tag{7}$$

In this case, $\pi=\pi(H_1)$, $\nu=\nu(H_1)$ *and* $\rho=\rho(H_1,H_2):=\operatorname{rank}[H_1\quad H_2]-\operatorname{rank} H_1$.

Proof. *Existence*. As H_1 is Hermitian, there exists a nonsingular matrix $P\in\mathbb{F}^{p\times p}$ such that $PH_1P^*=I_\pi\oplus(-I_\nu)\oplus 0$, where $\pi=\pi(H_1)$, $\nu=\nu(H_1)$. Suppose that $PH_2=[M_1^{\mathrm t}\quad M_2^{\mathrm t}\quad M_3^{\mathrm t}]^{\mathrm t}$, where $M_1\in\mathbb{F}^{\pi\times q}$, $M_2\in\mathbb{F}^{\nu\times q}$, $M_3\in\mathbb{F}^{(p-\pi-\nu)\times q}$. Note that $\operatorname{rank}M_3=\rho(H_1,H_2):=\operatorname{rank}[H_1\quad H_2]-\operatorname{rank}H_1$. Let $P_0\in\mathbb{F}^{(p-\pi-\nu)\times(p-\pi-\nu)}$, $Q\in\mathbb{F}^{q\times q}$ be nonsingular matrices such that $P_0M_3Q=I_\rho\oplus 0$, where $\rho=\rho(H_1,H_2)$. Then $[H_1\quad H_2]$ is block congruent to

$$\begin{bmatrix} I_\pi & 0 & 0 \\ 0 & I_\nu & 0 \\ 0 & 0 & P_0 \end{bmatrix} \left(P \begin{bmatrix} H_1 & H_2 \end{bmatrix} (P^* \oplus I_q)\right) \left[\begin{array}{ccc|c} I_\pi & 0 & 0 & -M_1 Q \\ 0 & I_\nu & 0 & M_2 Q \\ 0 & 0 & P_0^* & 0 \\ \hline 0 & 0 & 0 & Q \end{array}\right]$$

and this matrix has the prescribed form.

Unicity. If $[H_1 \quad H_2]$ is block congruent to a matrix of the form (7), it is easy to see that $\pi = \pi(H_1)$, $\nu = \nu(H_1)$ and $\rho = \operatorname{rank}[H_1 \quad H_2] - \operatorname{rank} H_1$. $\square$

Therefore, two matrices $[H_1 \quad H_2]$ and $[H_1' \quad H_2']$, where $H_1, H_1' \in \mathbb{F}^{p\times p}$ are Hermitian and $H_2, H_2' \in \mathbb{F}^{p\times q}$, are block congruent if and only if $\operatorname{In}(H_1) = \operatorname{In}(H_1')$ and $\rho(H_1, H_2) = \rho(H_1', H_2')$.

The main result of this section is the next theorem that can be viewed as giving a complete set of relations between the block similarity orbit of $[A \quad B]$ and the block congruence orbit of $[H_1 \quad H_2]$, when (4) is satisfied.

Theorem 7. *Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$. Let π, ν, δ and ρ be nonnegative integers such that $\pi + \nu + \delta = p$. The following are equivalent*:

(a_7) *There exists a Hermitian matrix $H_1 \in \mathbb{F}^{p\times p}$ and there exists $H_2 \in \mathbb{F}^{p\times q}$ such that* (4) *is satisfied*, $\operatorname{In}(H_1) = (\pi, \nu, \delta)$ *and* $\rho(H_1, H_2) = \rho$.
(b_7) $\rho = \delta \leqslant \operatorname{rank} B$ *and* $\operatorname{In}(A, B) \leqslant (\pi, \nu, 0)$.

Proof. (a_7) *implies* (b_7). If $\operatorname{rank}[H_1 \quad H_2] < p$, then $[H_1 \quad H_2]$ is block congruent to a matrix with its last row equal to zero. Without loss of generality, suppose that the last row of $[H_1 \quad H_2]$ is equal to zero. Then the entry (p, p) of the left-hand side of (4) is equal to zero, a contradiction. Therefore $\operatorname{rank}[H_1 \quad H_2] = p$. Hence $\rho = \delta$. Then $[H_1 \quad H_2]$ is block congruent to

$$\left[\begin{array}{cc|cc} \Delta & 0 & 0 & 0 \\ 0 & 0_\rho & I_\rho & 0 \end{array}\right],$$

where $\Delta \in \mathbb{F}^{(p-\rho)\times(p-\rho)}$ is Hermitian and $\operatorname{In}(\Delta) = (\pi, \nu, 0)$. Without loss of generality, assume that $[H_1 \quad H_2]$ has this form. Partition A and B accordingly:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \quad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix},$$

$A_{1,1} \in \mathbb{F}^{(p-\rho)\times(p-\rho)}$ and $B_{1,1} \in \mathbb{F}^{(p-\rho)\times\rho}$. Then $B_{2,1} + B_{2,1}^*$ is a principal submatrix of the left-hand side of (4). Then $B_{2,1} + B_{2,1}^* > 0$. Then $\operatorname{rank} B \geqslant \operatorname{rank} B_{2,1} = \rho = \delta$.

In this paragraph, we shall prove that $\operatorname{In}(A, B) \leqslant (\pi, \nu, 0)$. From now on, assume, without loss of generality, that $[A \quad B]$ has the form (6). Partition $[H_1 \quad H_2]$ accordingly:

$$\left[H_1 \mid H_2\right] = \left[\begin{array}{cc|c} G_{1,1} & G_{1,2} & G_{1,3} \\ G_{1,2}^* & G_{2,2} & G_{2,3} \end{array}\right],$$

where $G_{1,1} \in \mathbb{F}^{d\times d}$, $G_{2,2} \in \mathbb{F}^{(p-d)\times(p-d)}$. Then $NG_{1,1} + G_{1,1}N^*$ is a principal submatrix of the left-hand side of (4). Therefore $NG_{1,1} + G_{1,1}N^* > 0$. According to Theorem 4, $\delta(N) = 0$ and $\mathrm{In}(N) = \mathrm{In}(G_{1,1})$. From the Cauchy interlacing inequalities for the eigenvalues, it follows that $\pi(G_{1,1}) \leqslant \pi(H_1)$ and $\nu(G_{1,1}) \leqslant \nu(H_1)$. Then $\mathrm{In}(A, B) = \mathrm{In}(N) = \mathrm{In}(G_{1,1}) \leqslant (\pi, \nu, 0)$.

(b_7) *implies* (a_7). Let $\gamma_1|\cdots|\gamma_p$ be the invariant factors of (1). As $\rho \leqslant \operatorname{rank} B$, we have $\gamma_1 = \cdots = \gamma_\rho = 1$ and there exist nonsingular matrices $U \in \mathbb{F}^{p\times p}$, $V \in \mathbb{F}^{q\times q}$ such that $UBV = B_{1,1} \oplus I_\rho$, for some $B_{1,1} \in \mathbb{F}^{(p-\rho)\times(q-\rho)}$. Suppose that

$$\begin{bmatrix} A' & B' \end{bmatrix} := U\begin{bmatrix} A & B \end{bmatrix}(U^{-1} \oplus V) = \left[\begin{array}{cc|cc} A_{1,1} & A_{1,2} & B_{1,1} & 0 \\ A_{2,1} & A_{2,2} & 0 & I_\rho \end{array}\right],$$

where the blocks $A', A_{1,1}, A_{2,2}$ are square matrices. The invariant factors of $\left[xI_{p-\rho} - A_{1,1} \mid A_{1,2} \mid B_{1,1}\right]$ are $\gamma_{\rho+1}, \ldots, \gamma_p$. As $\mathrm{In}(\gamma_{\rho+1}\cdots\gamma_p) = \mathrm{In}(A, B) \leqslant (\pi, \nu, 0)$, choose a monic polynomial h such that $\mathrm{In}(\gamma_{\rho+1}\cdots\gamma_p h) = (\pi, \nu, 0)$. According to Lemma 1, there exist $X \in \mathbb{F}^{\rho\times(p-\rho)}$ and $Y \in \mathbb{F}^{(q-\rho)\times(p-\rho)}$ such that $A_{1,1} + A_{1,2}X + B_{1,1}Y$ has characteristic polynomial $\gamma_{\rho+1}\cdots\gamma_p h$. Then $[A \quad B]$ is block similar to

$$\begin{bmatrix} A'' & B'' \end{bmatrix} := \begin{bmatrix} I_{p-\rho} & 0 \\ -X & I_\rho \end{bmatrix}\begin{bmatrix} A' & B' \end{bmatrix}\left(\begin{bmatrix} I_{p-\rho} & 0 & 0 \\ X & I_\rho & 0 \\ Y & 0 & I_{q-\rho} \end{bmatrix} \oplus I_\rho\right)$$

and $[A'' \quad B'']$ has the form

$$\left[\begin{array}{cc|cc} A'_{1,1} & A_{1,2} & B_{1,1} & 0 \\ * & * & * & I_\rho \end{array}\right],$$

where $A'_{1,1} = A_{1,1} + A_{1,2}X + B_{1,1}Y$. Clearly $[A'' \quad B'']$ is block similar to

$$\left[\begin{array}{cc|cc} A'_{1,1} & A_{1,2} & B_{1,1} & 0 \\ 0 & 0 & 0 & I_\rho \end{array}\right].$$

Without loss of generality, assume that $[A \quad B]$ has this form. According to Theorem 4, there exists a Hermitian matrix $\varDelta \in \mathbb{F}^{(p-\rho)\times(p-\rho)}$ such that $A'_{1,1}\varDelta + \varDelta A'^*_{1,1} > 0$ and $\mathrm{In}(\varDelta) = \mathrm{In}(A'_{1,1}) = (\pi, \nu, 0)$. Let

$$H_1 = \begin{bmatrix} \varDelta & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{p\times p}, \quad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_\rho \end{bmatrix} \in \mathbb{F}^{p\times q}.$$

Then $\mathrm{In}(H_1) = (\pi, \nu, \delta)$, $\rho(H_1, H_2) = \rho$ and

$$AH_1 + H_1A^* + BH_2^* + H_2B^* = (A'_{1,1}\varDelta + \varDelta A'^*_{1,1}) \oplus 2I_\rho > 0. \qquad \square$$

3. A theorem by Chen and Wimmer

Chen [2] and Wimmer [11] proved that, with the notation of Theorem 4, if

(*e*) There exists a Hermitian matrix $H' \in \mathbb{F}^{n\times n}$, congruent to H, such that $K := LH' + H'L^* \geqslant 0$ and (L, K) is completely controllable, then (d_4) is satisfied. The converse is trivially true, as $(d_4) \Rightarrow (a_4) \Rightarrow (e)$. Our next purpose is to obtain a version of Chen's and Wimmer's theorem for pairs of matrices.

Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$. The controllability matrix of the pair (A, B) is the matrix

$$\mathscr{C}(A, B) := \begin{bmatrix} B & AB & \cdots & A^{p-1}B \end{bmatrix} \in \mathbb{F}^{p\times pq}.$$

Recall that the pair (A, B) is completely controllable if and only if all the invariant factors of (1) are equal to 1 if and only if

$$\min_{\lambda\in\mathbb{C}} \operatorname{rank} \left[\lambda I_p - A \;\middle|\; B\right] = p$$

if and only if $\operatorname{rank}\mathscr{C}(A, B) = p$. Moreover, if (6) is the canonical form for block similarity of $[A \quad B]$, then (M_1, M_2) is completely controllable.

Lemma 8. *Let $G, Y \in \mathbb{F}^{p\times p}$ be Hermitian matrices and suppose that Y is nonsingular. Then there exists a positive real number ϵ such that, for every $\lambda \geqslant \epsilon$,* $\operatorname{In}(G + \lambda Y) = \operatorname{In}(Y)$.

Proof. For every positive real number λ, $\operatorname{In}(G + \lambda Y) = \operatorname{In}(\lambda^{-1}G + Y)$. As $\lim_{\lambda\to+\infty}(\lambda^{-1}G) = 0$ and Y is nonsingular, the conclusion follows from the continuity of the eigenvalues. $\square$

Lemma 9. *Let $G_{1,1} \in \mathbb{F}^{d\times d}$, $Y \in \mathbb{F}^{(p-d)\times(p-d)}$ be Hermitian nonsingular matrices. Let $G_{1,2} \in \mathbb{F}^{d\times(p-d)}$. Then there exists a positive real number ϵ such that, for every $\lambda \geqslant \epsilon$,*

$$\operatorname{In}\begin{bmatrix} G_{1,1} & G_{1,2} \\ G_{1,2}^* & \lambda Y \end{bmatrix} = \operatorname{In}(G_{1,1}) + \operatorname{In}(Y).$$

Proof. For every positive real number λ,

$$\begin{bmatrix} G_{1,1} & G_{1,2} \\ G_{1,2}^* & \lambda Y \end{bmatrix}$$

is congruent to $G_{1,1} \oplus (\lambda Y - G_{1,2}^* G_{1,1}^{-1} G_{1,2})$. Then the conclusion follows from the previous lemma. $\square$

Lemma 10. *Let $M_1 \in \mathbb{F}^{p\times p}$, $M_2 \in \mathbb{F}^{p\times q}$. Suppose that (M_1, M_2) is completely controllable. Then there exist $Y_-, Y_+ \in \mathbb{F}^{p\times p}$, $Z_-, Z_+ \in \mathbb{F}^{p\times q}$ such that Y_-, Y_+ are Hermitian, $Y_- < 0$, $Y_+ > 0$ and*

$$M_1Y_- + Y_-M_1^* + M_2Z_-^* + Z_-M_2^* > 0, \tag{8}$$

$$M_1Y_+ + Y_+M_1^* + M_2Z_+^* + Z_+M_2^* > 0. \tag{9}$$

Proof. Note that we may assume, without loss of generality, that the matrix $[M_1 \quad M_2]$ is in the canonical form for block similarity. We shall prove that there exist $Y_- \in \mathbb{F}^{p\times p}$, $Z_- \in \mathbb{F}^{p\times q}$ such that $Y_- < 0$ and (8) is satisfied. The remaining of the proof is analogous.

First suppose that rank$M_2 = 1$. We may assume that

$$M_1 = C(x^p), \quad M_2 = \begin{bmatrix} e_p^{(p)} & 0 & \cdots & 0 \end{bmatrix}.$$

In this case, the proof is by induction on p. If $p = 1$, let

$$\left[\begin{array}{c|c} Y_- & Z_- \end{array}\right] := \left[\begin{array}{c|cccc} -1 & 1 & 0 & \cdots & 0 \end{array}\right].$$

Suppose that $p \geqslant 2$. Suppose that

$$\left[\begin{array}{c|c} M_1 & M_2 \end{array}\right] = \left[\begin{array}{cc|cccc} L_1 & L_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{array}\right],$$

where $L_1 \in \mathbb{F}^{(p-1)\times(p-1)}$, $L_2 \in \mathbb{F}^{(p-1)\times 1}$. According to the induction assumption, there exists a negative definite matrix $R \in \mathbb{F}^{(p-1)\times(p-1)}$ and there exists $S \in \mathbb{F}^{(p-1)\times 1}$ such that $L_1R + RL_1^* + L_2S^* + SL_2^* > 0$. Choose $\mu \in \mathbb{R}$ so that

$$Y_- := \begin{bmatrix} R & S \\ S^* & \mu \end{bmatrix}$$

has determinant $(-1)^p$. Then choose $\lambda \in \mathbb{R}$ so that, with

$$Z_- := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \lambda & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{F}^{p\times q},$$

the matrix

$$\begin{aligned} &M_1Y_- + Y_-M_1^* + M_2Z_-^* + Z_-M_2^* \\ &\quad = \begin{bmatrix} L_1R + RL_1^* + L_2S^* + SL_2^* & L_1S + \mu L_2 \\ S^*L_1^* + \mu L_2^* & 2\lambda \end{bmatrix} \end{aligned}$$

has positive determinant. Using the criteria of the determinants of the leading principal minors, we deduce that Y_- is negative definite and $M_1Y_- + Y_-M_1^* + M_2Z_-^* + Z_-M_2^*$ is positive definite.

Now we suppose that $u :=$ rank$M_2 \geqslant 2$. We may assume that

$$\begin{aligned} M_1 &= C(x^{\mu_1}) \oplus \cdots \oplus C(x^{\mu_u}), \\ M_2 &= \begin{bmatrix} e_{\mu_1}^{(p)} & e_{\mu_1+\mu_2}^{(p)} & \cdots & e_{\mu_1+\cdots+\mu_u}^{(p)} & 0 \end{bmatrix}. \end{aligned}$$

According to the previous case, for every $i \in \{1, \ldots, u\}$, there exists a negative definite matrix $Y_i \in \mathbb{F}^{\mu_i\times\mu_i}$ and there exists $Z_i \in \mathbb{F}^{\mu_i\times 1}$ such that

$$C(x^{\mu_i})Y_i + Y_iC(x^{\mu_i})^* + e_{\mu_i}^{(\mu_i)}Z_i^* + Z_ie_{\mu_i}^{(\mu_i)*} > 0.$$

Let $Y_- = Y_1 \oplus \cdots \oplus Y_u \in \mathbb{F}^{p\times p}$ and $Z_- = [Z_1 \oplus \cdots \oplus Z_u \quad 0] \in \mathbb{F}^{p\times q}$. Then Y_- is negative definite and (8) is satisfied. $\square$

Lemma 11. *Let $M_1 \in \mathbb{F}^{p'\times p'}$, $M_2 \in \mathbb{F}^{p'\times q}$. Suppose that (M_1, M_2) is completely controllable. Then, for every $c_1, \ldots, c_{p'} \in \mathbb{F}$, $[M_1 \quad M_2]$ is block similar to a matrix* *of the form* $[M_1' \quad M_2']$, *where* $M_1' \in \mathbb{F}^{p' \times p'}$ *is upper triangular with its entry* (i,i) *equal to* c_i, *for every* $i \in \{1,\dots,p'\}$, *and* M_2' *has the form*

$$\begin{bmatrix} 0 & 0 \\ I_{\operatorname{rank} M_2} & 0 \end{bmatrix}. \tag{10}$$

Proof. By induction on p'. As (M_1, M_2) is completely controllable, $\operatorname{rank} M_2 > 0$. Let $P \in \mathbb{F}^{p' \times p'}$ and $Q \in \mathbb{F}^{q \times q}$ be nonsingular matrices such that $M_2' := P M_2 Q$ has the form (10). Let $s = \operatorname{rank} M_2$.

If $p' = s$, then, for every $M_1' \in \mathbb{F}^{p' \times p'}$, $[M_1 \quad M_2]$ is block similar to $[M_1' \quad M_2']$ and the result is trivial.

Now suppose that $p' > s$. Then $[M_1 \quad M_2]$ is block similar to

$$P\begin{bmatrix} M_1 & M_2 \end{bmatrix}(P^{-1} \oplus Q) = \left[\begin{array}{cc|cc} N_1 & N_2 & 0 & 0 \\ * & * & I_s & 0 \end{array}\right],$$

where $N_1 \in \mathbb{F}^{(p'-s)\times(p'-s)}$, $N_2 \in \mathbb{F}^{(p'-s)\times s}$. Note that (N_1, N_2) is completely controllable. According to the induction assumption, $[N_1 \quad N_2]$ is block similar to a matrix of the form $[N_1' \quad N_2']$, where $N_1' \in \mathbb{F}^{(p'-s)\times(p'-s)}$ is upper triangular with its entry (i,i) equal to c_i, for every $i \in \{1,\dots,p'-s\}$. Then $[M_1 \quad M_2]$ is block similar to

$$\left[\begin{array}{cc|cc} N_1' & N_2' & 0 & 0 \\ 0 & \operatorname{diag}(c_{p'-s+1},\dots,c_{p'}) & I_s & 0 \end{array}\right].$$

The last matrix has the prescribed form. $\square$

The following result generalizes Chen's and Wimmer's theorem to pairs of matrices.

Theorem 12. *Let* $A \in \mathbb{F}^{p \times p}$, $B \in \mathbb{F}^{p \times q}$. *Let* π, ν, δ *and* ρ *be nonnegative integers such that* $\pi + \nu + \delta = p$. *Let* $\gamma_1 | \cdots | \gamma_p$ *be the invariant factors of* (1). *The following are equivalent*:

(a_{12}) *There exists a Hermitian matrix* $H_1 \in \mathbb{F}^{p \times p}$ *and there exists* $H_2 \in \mathbb{F}^{p \times q}$ *such that* $\operatorname{In}(H_1) = (\pi, \nu, \delta)$, $\rho(H_1, H_2) = \rho$,

$$K := AH_1 + H_1A^* + BH_2^* + H_2B^* \geqslant 0, \tag{11}$$

and $(A, [B \quad K])$ *is completely controllable.*

(b_{12}) $\rho \leqslant q$, $\rho \leqslant \delta \leqslant p - \deg(\gamma_1 \cdots \gamma_p)$ *and* $\operatorname{In}(A, B) \leqslant (\pi, \nu, 0)$.

Proof. (a_{12}) *implies* (b_{12}). First suppose that H_1 is nonsingular. Clearly $\rho = \delta = 0 \leqslant p - \deg(\gamma_1 \cdots \gamma_p)$. Without loss of generality, suppose that

$$H_1 = I_\pi \oplus (-I_\nu), \quad H_2 = 0.$$

Partition $[A \quad B]$ accordingly:

$$\begin{bmatrix} A & | & B \end{bmatrix} = \left[\begin{array}{cc|c} A_{1,1} & A_{1,2} & B_1 \\ A_{2,1} & A_{2,2} & B_2 \end{array}\right],$$

where $A_{1,1} \in \mathbb{F}^{\pi\times\pi}$, $A_{2,2} \in \mathbb{F}^{\nu\times\nu}$. Let $j \in \{-1, 1\}$. Let

$$L_j = \begin{bmatrix} A_{1,1} & A_{1,2} & B_1 \\ A_{2,1} & A_{2,2} & B_2 \\ -jB_1^* & jB_2^* & jI_q \end{bmatrix}, \quad G_j = \begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & jI_q \end{bmatrix}.$$

Then

$$L_jG_j + G_jL_j^* = K \oplus 2I_q \geqslant 0.$$

As $(A, [B \quad K])$ is completely controllable, it follows that $(L_j, L_jG_j + G_jL_j^*)$ is completely controllable. According to Chen's [2] and Wimmer's [11] theorem, $\mathrm{In}(L_j) = \mathrm{In}(G_j)$. According to Lemma 2, $\mathrm{In}(A, B) \leqslant \mathrm{In}(L_j)$. Then

$$\mathrm{In}(A, B) \leqslant \mathrm{In}(L_{-1}) = \mathrm{In}(G_{-1}) = (\pi, \nu + q, 0),$$
$$\mathrm{In}(A, B) \leqslant \mathrm{In}(L_1) = \mathrm{In}(G_1) = (\pi + q, \nu, 0).$$

Therefore $\mathrm{In}(A, B) \leqslant (\pi, \nu, 0)$.

Now we shall give the proof for the general case. The condition $\rho \leqslant q$ is trivial. Without loss of generality, suppose that $[A \quad B]$ has the form (6). Partition $[H_1 \quad H_2]$ accordingly:

$$\begin{bmatrix} H_1 & | & H_2 \end{bmatrix} = \left[\begin{array}{cc|c} G_{1,1} & G_{1,2} & G_{1,3} \\ G_{1,2}^* & G_{2,2} & G_{2,3} \end{array}\right],$$

where $G_{1,1} \in \mathbb{F}^{d\times d}$, $G_{2,2} \in \mathbb{F}^{(p-d)\times(p-d)}$. Then

$$K = \begin{bmatrix} NG_{1,1} + G_{1,1}N^* & NG_{1,2} + G_{1,2}M_1^* + G_{1,3}M_2^* \\ G_{1,2}^*N^* + M_1G_{1,2}^* + M_2G_{1,3}^* & M_1G_{2,2} + G_{2,2}M_1^* + M_2G_{2,3}^* + G_{2,3}M_2^* \end{bmatrix}. \tag{12}$$

In order to get a contradiction, suppose that $G_{1,1}$ is singular. Let $e = \mathrm{rank}G_{1,1}$. Then there exists a nonsingular matrix $U \in \mathbb{F}^{d\times d}$ such that $G'_{1,1} := UG_{1,1}U^* = 0_{d-e} \oplus \varDelta$, where $\varDelta$ is nonsingular or vanishes (i.e., $G_{1,1} = 0$). Partition $N' := UNU^{-1}$ accordingly:

$$N' = \begin{bmatrix} N_{1,1} & N_{1,2} \\ N_{2,1} & N_{2,2} \end{bmatrix},$$

where $N_{1,1} \in \mathbb{F}^{(d-e)\times(d-e)}$. Then $NG_{1,1} + G_{1,1}N^*$ is congruent to

$$N'G'_{1,1} + G'_{1,1}N'^* = \begin{bmatrix} 0_{d-e} & N_{1,2}\varDelta \\ \varDelta N_{1,2}^* & N_{2,2}\varDelta + \varDelta N_{2,2}^* \end{bmatrix}.$$

As $NG_{1,1} + G_{1,1}N^* \geqslant 0$, we deduce that $N_{1,2}\varDelta = 0$. As $\varDelta$ is nonsingular, $N_{1,2} = 0$. Note that $N'G'_{1,1} + G'_{1,1}N'^*$ is a leading principal submatrix of

$$K' := (U \oplus I_{p-d})K(U^* \oplus I_{p-d}).$$

As $K \geqslant 0$ and K' is congruent to K, we deduce that the first $d-e$ rows of K' are zero. Then $[A \quad B \quad K]$ is block similar to

$$(U \oplus I_{p-d})\begin{bmatrix}A & B & K\end{bmatrix}(U^{-1} \oplus I_{p-d+q} \oplus U^* \oplus I_{p-d}) = \begin{bmatrix}N_{1,1} & 0 & 0\\ * & * & *\end{bmatrix}.$$

If $\lambda \in \mathbb{C}$ is an eigenvalue of $N_{1,1}$, then $\operatorname{rank}\begin{bmatrix}\lambda I_p - A & B & K\end{bmatrix} < p$. This is a contradiction, because $(A, [B \quad K])$ is completely controllable. Therefore $G_{1,1}$ is nonsingular. Then $d = \operatorname{rank} G_{1,1} \leqslant \operatorname{rank} H_1 = p - \delta$ and $\rho \leqslant \delta \leqslant p - d = p - \deg(\gamma_1 \cdots \gamma_p)$.

For every Hermitian matrix $Y \in \mathbb{F}^{(p-d)\times(p-d)}$ and every $Z \in \mathbb{F}^{(p-d)\times q}$, let $K_{Y,Z}$ be the matrix that results from K on replacing $G_{2,2}$ by Y and $G_{2,3}$ by Z in (12).

As (M_1, M_2) is completely controllable, all the invariant factors of

$$\begin{bmatrix}xI_{p-d} - M_1 & M_2\end{bmatrix} \tag{13}$$

are equal to 1 and the Smith canonical form of (13) is $[I_{p-d} \quad 0]$. Bearing in mind the form (6) of $[A \quad B]$, we deduce that

$$\begin{bmatrix}xI_p - A & B & K_{Y,Z}\end{bmatrix}$$

is equivalent, over $F[x]$, to

$$\begin{bmatrix}xI_d - N & NG_{1,1} + G_{1,1}N^* & NG_{1,2} + G_{1,2}M_1^* + G_{1,3}M_2^*\end{bmatrix} \oplus \begin{bmatrix}I_{p-d} & 0\end{bmatrix}. \tag{14}$$

Note that (14) does not depend on Y and Z. In particular, we know that $(A, [B \quad K])$ is completely controllable and, therefore, (14) has all its invariant factors equal to 1. Consequentely $(A, [B \quad K_{Y,Z}])$ is completely controllable, for every Hermitian matrix $Y \in \mathbb{F}^{(p-d)\times(p-d)}$ and every $Z \in \mathbb{F}^{(p-d)\times q}$.

According to Lemma 10, there exist $Y_-, Y_+ \in \mathbb{F}^{(p-d)\times(p-d)}$, $Z_-, Z_+ \in \mathbb{F}^{(p-d)\times q}$ such that Y_-, Y_+ are Hermitian, $Y_- < 0$, $Y_+ > 0$ and (8), (9) are satisfied. For every positive real number λ, let

$$H_{1,-\lambda} = \begin{bmatrix}G_{1,1} & G_{1,2}\\ G_{1,2}^* & \lambda Y_-\end{bmatrix}, \quad H_{2,-\lambda} = \begin{bmatrix}G_{1,3}\\ \lambda Z_-\end{bmatrix},$$
$$H_{1,\lambda} = \begin{bmatrix}G_{1,1} & G_{1,2}\\ G_{1,2}^* & \lambda Y_+\end{bmatrix}, \quad H_{2,\lambda} = \begin{bmatrix}G_{1,3}\\ \lambda Z_+\end{bmatrix}.$$

Note that

$$AH_{1,-\lambda} + H_{1,-\lambda}A^* + BH_{2,-\lambda}^* + H_{2,-\lambda}B^* = K_{\lambda Y_-,\lambda Z_-},$$
$$AH_{1,\lambda} + H_{1,\lambda}A^* + BH_{2,\lambda}^* + H_{2,\lambda}B^* = K_{\lambda Y_+,\lambda Z_+}.$$

According to Lemma 9, there exists a positive real number ϵ such that, for every $\lambda \geqslant \epsilon$,

$$\mathrm{In}(H_{1,-\lambda}) = \mathrm{In}(G_{1,1}) + (0, p-d, 0),$$
$$\mathrm{In}(H_{1,\lambda}) = \mathrm{In}(G_{1,1}) + (p-d, 0, 0),$$

and $H_{1,-\lambda}$ and $H_{1,\lambda}$ are nonsingular.

Let $V \in \mathbb{F}^{d\times d}$ be a nonsingular matrix such that

$$V(NG_{1,1} + G_{1,1}N^*)V^* = 0_{d-r} \oplus \Delta,$$

where $r = \mathrm{rank}(NG_{1,1} + G_{1,1}N^*)$ and $\Delta \in \mathbb{F}^{r\times r}$ is positive definite. Then $(V \oplus I_{p-d})K(V^* \oplus I_{p-d})$ has the form

$$\begin{bmatrix} 0_{d-r} & 0 & P \\ 0 & \Delta & Q \\ P^* & Q^* & M_1G_{2,2} + G_{2,2}M_1^* + M_2G_{2,3}^* + G_{2,3}M_2^* \end{bmatrix},$$

for some matrices P, Q. As $K \geqslant 0$, we deduce that $P = 0$. Note that, for every positive real number λ, $(V \oplus I_{p-d})K_{\lambda Y_-,\lambda Z_-}(V^* \oplus I_{p-d})$ has the form

$$0_{d-r} \oplus \begin{bmatrix} \Delta & Q \\ Q^* & \lambda(M_1Y_- + Y_-M_1^* + M_2Z_-^* + Z_-M_2^*) \end{bmatrix}.$$

It follows from Lemma 9 that there exists a positive real number ϵ_- such that, for every $\lambda \geqslant \epsilon_-$, $K_{\lambda Y_-,\lambda Z_-}$ is positive semidefinite. Analogously, there exists a positive real number ϵ_+ such that, for every $\lambda \geqslant \epsilon_+$, $K_{\lambda Y_+,\lambda Z_+}$ is positive semidefinite.

For $\lambda \geqslant \max\{\epsilon, \epsilon_-, \epsilon_+\}$, it follows, from the previously studied case and the Cauchy interlacing inequalities, that

$$\mathrm{In}(A, B) \leqslant \mathrm{In}(H_{1,-\lambda}) = \mathrm{In}(G_{1,1}) + (0, p-d, 0) \leqslant (\pi, \nu + p - d, 0),$$
$$\mathrm{In}(A, B) \leqslant \mathrm{In}(H_{1,\lambda}) = \mathrm{In}(G_{1,1}) + (p-d, 0, 0) \leqslant (\pi + p - d, \nu, 0).$$

Therefore $\mathrm{In}(A, B) \leqslant (\pi, \nu, 0)$.

(b_{12}) *implies* (a_{12}). Suppose that (6) is the canonical form for block similarity of $[A \quad B]$. We have $\mathrm{In}(N) = \mathrm{In}(A, B) \leqslant (\pi, \nu, 0)$. Then $d = \deg(\gamma_1 \cdots \gamma_p) \leqslant \pi + \nu$. Let $p' = p - d$. Let $[M_1' \quad M_2']$ be a matrix, block similar to $[M_1 \quad M_2]$, of the form indicated in Lemma 11, with the elements $c_1, \ldots, c_{p'}$ chosen so that, if M_0 is the leading principal submatrix of M_1' of size $(\pi + \nu - d) \times (\pi + \nu - d)$, then $\mathrm{In}(N \oplus M_0) = (\pi, \nu, 0)$.

Then $[A \quad B]$ is block similar to

$$\left[\begin{array}{cc|c} N & 0 & 0 \\ 0 & M_1' & M_2' \end{array}\right].$$

As $\rho \leqslant \delta = p - \pi - \nu$, it is not hard to deduce that there exists a permutation matrix $Q \in \mathbb{F}^{q\times q}$ such that

$$M_2'Q = \begin{bmatrix} 0 & B_{1,2}' \\ B_{2,1} & B_{2,2} \end{bmatrix} \in \mathbb{F}^{(p-d)\times q},$$

where

$$B_{2,1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} \in \mathbb{F}^{\delta\times\rho},$$

$r = \min\{\operatorname{rank} B, \rho\}$. Partition $N \oplus M_1'$ as

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, \quad \text{where } A_{1,1} = N \oplus M_0 \in \mathbb{F}^{(\pi+\nu)\times(\pi+\nu)}.$$

Then $[A \quad B]$ is block similar to

$$\left[\begin{array}{cc|cc} A_{1,1} & A_{1,2} & 0 & B_{1,2} \\ 0 & A_{2,2} & B_{2,1} & B_{2,2} \end{array}\right], \quad \text{where } B_{1,2} = \begin{bmatrix} 0 \\ B_{1,2}' \end{bmatrix}. \tag{15}$$

Without loss of generality, suppose that $[A \quad B]$ has the form (15).

According to Theorem 4, there exists a Hermitian matrix $\varDelta \in \mathbb{F}^{(\pi+\nu)\times(\pi+\nu)}$ such that $A_{1,1}\varDelta + \varDelta A_{1,1}^* > 0$ and $\operatorname{In}(\varDelta) = \operatorname{In}(A_{1,1}) = \operatorname{In}(N \oplus M_0) = (\pi, \nu, 0)$. Let

$$\left[H_1 \mid H_2\right] := \left[\begin{array}{ccc|cc} \varDelta & 0 & 0 & 0 & 0 \\ 0 & 0_{\delta-\rho} & 0 & 0 & 0 \\ 0 & 0 & 0_\rho & I_\rho & 0 \end{array}\right],$$

where H_1 is of size $p \times p$ and H_2 is of size $p \times q$. Then $\operatorname{In}(H_1) = (\pi, \nu, \delta)$, $\rho(H_1, H_2) = \rho$,

$$K := AH_1 + H_1A^* + BH_2^* + H_2B^* = (A_{1,1}\varDelta + \varDelta A_{1,1}^*) \oplus 0 \oplus 2I_r \geqslant 0.$$

and $(A, [B \quad K])$ is completely controllable. □

4. On a inertia theorem by Loewy

For any $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$, denote the number of nonconstant invariant factors of (1) by $i(A, B)$. Recall that (A, B) is completely controllable if and only if $i(A, B) = 0$. The following lemma is a simple corollary of [6,9], as we show. It is also a consequence of [1]; and a consequence of [13], when $q = 0$.

Lemma 13. *Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$. Then $i(A, B)$ is the smallest nonnegative integer τ for which there exists $X \in \mathbb{F}^{p\times\tau}$ such that $(A, [B \quad X])$ is completely controllable.* [*The result is valid when $q = 0$.*]

Proof. Let $\tau = i(A, B)$. According to [6,9], there exists $M(x) \in \mathbb{F}[x]^{p\times\tau}$ such that $\left[xI_p - A \mid B \mid M(x)\right]$ has all its invariant factors equal to 1. Suppose that $M(x) = (xI_p - A)Q(x) + X$, where $Q(x) \in \mathbb{F}[x]^{p\times\tau}$, $X \in \mathbb{F}^{p\times\tau}$. Then $\left[xI_p - A \mid B \mid M(x)\right]$ and $\left[xI_p - A \mid B \mid X\right]$ are equivalent, and, therefore, $(A, [B \quad X])$ is completely controllable.

Now suppose that $Y \in \mathbb{F}^{p\times\sigma}$, with $\sigma < \tau$. Let $\gamma_1|\cdots|\gamma_p$ be the invariant factors of (1) and let $\eta_1|\cdots|\eta_p$ be the invariant factors of $\left[xI_p - A \;\middle|\; B \;\middle|\; Y\right]$. According to [6,9], $\gamma_{p-\sigma}|\eta_p$. As $\gamma_{p-\sigma}$ is nonconstant, it follows that $(A, [B \quad Y])$ is not completely controllable. $\square$

Loewy [4] has obtained necessary conditions for the statement that results from (e), when "(L, K) is completely controllable" is replaced by the more general assumption "$\operatorname{rank}\mathscr{C}(L, K) = l$".

The next theorem gives a necessary condition for the statement that results from (e), when "(L, K) is completely controllable" is replaced by the more general assumption "$i(L, K) = \tau$". We also give a version of this result for pairs of matrices.

Theorem 14. *Let $L \in \mathbb{F}^{n\times n}$. Let π, ν, δ, τ be nonnegative integers such that $\pi + \nu + \delta = n$. Let $\alpha_1|\cdots|\alpha_n$ be the invariant factors of $xI_n - L$. If there exists a Hermitian matrix $H \in \mathbb{F}^{n\times n}$ such that* $\operatorname{In}(H) = (\pi, \nu, \delta)$, $K := LH + HL^* \geqslant 0$ *and* $i(L, K) = \tau$, *then*

(b_{14}) $\operatorname{In}(\alpha_1\cdots\alpha_{n-\tau}) \leqslant (\pi, \nu, 0)$. [*With the convention that* $\alpha_1\cdots\alpha_{n-\tau} = 1$, *when* $n = \tau$.]

Remark. The condition (b_{14}) implies that $\deg(\alpha_1\cdots\alpha_{n-\tau}) \leqslant \pi + \nu = n - \delta$ and, therefore, $\delta \leqslant n - \deg(\alpha_1\cdots\alpha_{n-\tau})$. As $\deg(\alpha_1\cdots\alpha_n) = n$, it is easy to deduce that, when $\tau = 0$, (b_{14}) is equivalent to (d_4).

We omit the proof of Theorem 14, because it is analogous to the proof of the next theorem.

Theorem 15. *Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$. Let π, ν, δ, τ be nonnegative integers such that $\pi + \nu + \delta = p$. Let $\gamma_1|\cdots|\gamma_p$ be the invariant factors of* (1). *If there exists a Hermitian matrix $H_1 \in \mathbb{F}^{p\times p}$ and there exists $H_2 \in \mathbb{F}^{p\times q}$ such that* $\operatorname{In}(H_1) = (\pi, \nu, \delta)$, (11) *holds and* $i(A, [B \quad K]) = \tau$, *then* $\operatorname{In}(\gamma_1\cdots\gamma_{p-\tau}) \leqslant (\pi, \nu, 0)$. [*With the convention that* $\gamma_1\cdots\gamma_{p-\tau} = 1$, *when* $p = \tau$.]

Proof. Bearing in mind the canonical form for block similarity of $[A \quad B \quad K]$ (cf. Lemma 5), it follows that there exists a nonsingular matrix $P \in \mathbb{F}^{p\times p}$ such that

$$A' := PAP^{-1} = \begin{bmatrix} N & 0 \\ M_0 & M_1 \end{bmatrix}, \quad B' := PB = \begin{bmatrix} 0 \\ M_2 \end{bmatrix},$$
$$K' := PKP^* = \begin{bmatrix} 0 & 0 \\ 0 & M_3 \end{bmatrix},$$

where $N \in \mathbb{F}^{e\times e}$, $M_1 \in \mathbb{F}^{(p-e)\times(p-e)}$, $M_2 \in \mathbb{F}^{(p-e)\times q}$, $M_3 \in \mathbb{F}^{(p-e)\times(p-e)}$, $(M_1, [M_2 \quad M_3])$ is completely controllable and the number of nonconstant invariant factors of $xI_e - N$ is equal to $i(A, [B \quad K]) = \tau$.

According to Lemma 13, there exists $D \in \mathbb{F}^{e\times\tau}$ such that (N, D) is completely controllable. Then $(A', [B' \quad E' \quad K'])$ is completely controllable, where

$$E' := \begin{bmatrix} D \\ 0 \end{bmatrix}.$$

Moreover, $K' = A'H_1' + H_1'A'^* + B'H_2'^* + H_2'B'^* \geqslant 0$, where $H_1' = PH_1P^*$ and $H_2' = PH_2$. According to Theorem 12, $\delta(H_1) \leqslant p - \deg(\eta_1 \cdots \eta_p)$, where $\eta_1|\cdots|\eta_p$ are the invariant factors of $\left[xI_p - A' \;\middle|\; B' \;\middle|\; E'\right]$, and $\mathrm{In}(A', [B' \quad E']) \leqslant (\pi(H_1), \nu(H_1), 0)$. From the interlacing inequalities for the invariant factors [6,9], $\eta_i|\gamma_i$ and $\gamma_i|\eta_{i+\tau}$, for every significant index i. Therefore

$$\mathrm{In}(\gamma_1 \cdots \gamma_{p-\tau}) \leqslant \mathrm{In}(\eta_1 \cdots \eta_p) = \mathrm{In}\left(A', \left[B' \quad E'\right]\right) \leqslant (\pi(H_1), \nu(H_1), 0).$$

□

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